

INTEGRABLE SOLUTIONS OF INHOMOGENEOUS REFINEMENT TYPE EQUATIONS ON INTERVALS

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ABSTRACT. Given a probability measure P on a σ -algebra of subsets of a set Ω , an interval $I \subset \mathbb{R}$, $g \in L^1(I)$, and a function $\varphi: I \times \Omega \rightarrow I$ fulfilling some conditions we obtain results on the existence of solutions $f \in L^1(I)$ of the inhomogeneous refinement type equation

$$f(x) = \int_{\Omega} |\varphi'_x(x, \omega)| f(\varphi(x, \omega)) dP(\omega) + g(x).$$

1. INTRODUCTION

A discrete inhomogeneous refinement equation

$$f(x) = \sum_{n \in \mathbb{Z}} c_n f(kx - n) + g(x)$$

as well as its continuous counterpart

$$f(x) = \int_{\mathbb{R}} f(kx - y) d\mu(y) + g(x)$$

were studied in many papers. For example, the discrete inhomogeneous refinement equation has been used in [7] for a construction of multiwavelets from a "fractal equation", in [16] for a construction of boundary wavelets, in [17] for the study of convergence of cascade algorithms and in [9] for the study of quad/triangular subdivision. A complete uniform characterization of the existence of distributional solutions of both the above classical inhomogeneous refinement equations was obtained in [8] to cover all cases of interest.

A common extension of both the above inhomogeneous refinement equations is the following inhomogeneous refinement type equation

$$(1) \quad f(x) = \int_{\Omega} |\varphi'_x(x, \omega)| f(\varphi(x, \omega)) P(d\omega) + g(x).$$

Integrable solutions of the poly-scale version of equation (1) have been recently investigated by the use of the Banach fixed point theorem in [12]. In the present paper we are interested in integrable solutions $f: I \rightarrow \mathbb{R}$ of equation (1) assuming that (Ω, \mathcal{A}, P) is a complete probability space, $I \subset \mathbb{R}$ is an open interval (finite or infinite; possibly equals to the whole real line), $g \in L^1(I)$ and $\varphi: I \times \Omega \rightarrow I$ is a function satisfying the following conditions:

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- (a) $\varphi(\cdot, \omega)$ is a diffeomorphism from I onto I for every $\omega \in \Omega$;
- (b) $\varphi(x, \cdot)$ is \mathcal{A} -measurable for every $x \in I$;
- (c) $(\lambda \otimes P)(\varphi^{-1}(B)) = 0$ for every Borel set $B \in \mathcal{B}(I)$ of Lebesgue measure λ equals zero.

The reader may ask why we choose the interval I to be open. No special reason. In fact all the presented below results can be reformulated if we start with I to be closed or closed on one side.

2. PRELIMINARIES

We begin by observing that if $g: I \rightarrow \mathbb{R}$ is a representative of $g \in L^1(I)$ and $f, \tilde{f}: I \rightarrow \mathbb{R}$ are two representatives of a function from $L^1(I)$ such that (1) holds for almost all $x \in I$, then \tilde{f} also satisfies (1) for almost all $x \in I$; see [10] for details in the homogeneous case of equation (1) with $I = \mathbb{R}$. This observation allows us to accept the following definition: A function $f \in L^1(I)$ is said to be an $L^1(I)$ -solution of equation (1), if every representative of f satisfies (1) for almost all $x \in I$ with respect to the Lebesgue measure.

Let X be a separable metric space. We say that $\psi: X \times \Omega \rightarrow X$ is a random-valued function, if it is measurable with respect to the product σ -algebra $\mathcal{B}(X) \otimes \mathcal{A}$. Following [4], we define the sequence $(\psi^n)_{n \in \mathbb{N}}$ of iterates of a random-valued function $\psi: X \times \Omega \rightarrow X$ as follows:

$$\psi^1(x, \varpi) = \psi(x, \omega_1) \quad \text{and} \quad \psi^{n+1}(x, \varpi) = \psi(\psi^n(x, \varpi), \omega_{n+1})$$

for all $x \in X$ and $\varpi = (\omega_1, \omega_2, \dots) \in \Omega^\infty$. Since $\psi^n(\cdot, \varpi)$ depends only on the first n coordinates of $\varpi \in \Omega^\infty$, we may consider the iterate ψ^n as a function defined on $X \times \Omega^\infty$ or, alternatively, on $X \times \Omega^n$. The basic property of iterates of rv-functions says that they are random-valued functions on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$.

It turns out that the above defined iterates form a random dynamical system (see [1]) and a Markov chain (see [15]). Moreover, iteration is the fundamental technique for solving functional equations, and iterates usually appear in the formulas for solutions (see [14]). In this paper we will apply a result on the convergence in law of the sequence of iterates of a random-valued function obtained in [2].

3. COMPACTLY SUPPORTED $L^1(\mathbb{R})$ -SOLUTIONS

In many applications compactly supported solutions of inhomogeneous, as well as homogeneous, refinement equations play an important role. Compactly supported distributional solutions of inhomogeneous refinement equations were considered, among others, in [5, 6, 18]. In this section we are interested in compactly supported $L^1(\mathbb{R})$ -solutions of equation (1). We begin with the following useful lemma.

Lemma 3.1. *Let $J \subset \mathbb{R}$ be an open interval and let $\alpha: J \rightarrow I$ be a diffeomorphism (onto I). Then:*

- (i) *The function $\phi: J \times \Omega \rightarrow J$ given by*

$$(2) \quad \phi(x, \omega) = \alpha^{-1}(\varphi(\alpha(x), \omega))$$

satisfies conditions (a)–(c) with I replaced by J ;

- (ii) If f is an $L^1(I)$ -solution of equation (1), then $\tilde{f} = |\alpha'| \cdot f \circ \alpha$ is an $L^1(J)$ -solution of the equation

$$(3) \quad \tilde{f}(x) = \int_{\Omega} |\phi'_x(x, \omega)| \tilde{f}(\phi(x, \omega)) P(d\omega) + \tilde{g}(x),$$

where $\tilde{g} = |\alpha'| \cdot g \circ \alpha$.

Proof. Assertion (i) is clear. To prove assertion (ii) assume that f satisfies (1). Clearly, $\tilde{f}, \tilde{g} \in L^1(J)$. Moreover,

$$\begin{aligned} \tilde{f}(x) &= |\alpha'(x)| f(\alpha(x)) = \int_{\Omega} |\varphi'_x(\alpha(x), \omega)| |\alpha'(x)| f(\varphi(\alpha(x), \omega)) P(d\omega) + |\alpha'(x)| g(\alpha(x)) \\ &= \int_{\Omega} |\phi'_x(x, \omega)| |\alpha'(\phi(x, \omega))| f(\alpha(\phi(x, \omega))) P(d\omega) + \tilde{g}(x) \\ &= \int_{\Omega} |\phi'_x(x, \omega)| \tilde{f}(\phi(x, \omega)) P(d\omega) + \tilde{g}(x), \end{aligned}$$

and the proof is complete. \square

Now we are in a position to prove the following result.

Theorem 3.2. Assume $I = \mathbb{R}$. Let $F \subset \mathbb{R}$ be a closed and non-degenerated interval such that

$$(4) \quad \text{supp } g \subset F$$

and

$$(5) \quad \varphi(\text{int} F \times \{\omega\}) = \text{int} F \quad \text{for almost all } \omega \in \Omega.$$

Then equation (1) has a compactly supported $L^1(\mathbb{R})$ -solution with $\text{supp } f \subset F$ if and only if there exists a diffeomorphism $\alpha: \text{int} F \rightarrow \mathbb{R}$ such that equation (3) with $\phi: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ given by (2) has an $L^1(\mathbb{R})$ -solution.

Proof. Assume that f is an $L^1(\mathbb{R})$ -solution of equation (1) with $\text{supp } f \subset F$.

Clearly, $f|_{\text{int} F} \in L^1(\text{int} F)$. Moreover, for almost all $x \in \text{int} F$ we have

$$\begin{aligned} f|_{\text{int} F}(x) &= f(x) = \int_{\Omega} |\varphi'_x(x, \omega)| f(\varphi(x, \omega)) P(d\omega) + g(x) \\ &= \int_{\Omega} |\varphi'_x(x, \omega)| f|_{\text{int} F}(\varphi(x, \omega)) P(d\omega) + g|_{\text{int} F}(x), \end{aligned}$$

which shows that $f|_{\text{int} F}$ is an $L^1(\text{int} F)$ -solution of equation (1). Finally, applying Lemma 3.1 with $I = \text{int} F$ and $J = \mathbb{R}$ we infer that equation (3), with $\phi: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ given by (2), has an $L^1(\mathbb{R})$ -solution.

To prove the converse implication assume that there is a diffeomorphism $\alpha: \text{int} F \rightarrow \mathbb{R}$ such that equation (3), with $\phi: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ given by (2), has an $L^1(\mathbb{R})$ -solution.

Using Lemma 3.1 with $I = \text{int}F$ and $J = \mathbb{R}$ once again we conclude that equation (1) has an $L^1(\text{int}F)$ -solution \tilde{f} .

Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ by putting

$$f(x) = \begin{cases} \tilde{f}(x) & \text{for } x \in \text{int}F, \\ 0 & \text{for } x \in \mathbb{R} \setminus \text{int}F. \end{cases}$$

Clearly, $f \in L^1(\mathbb{R})$, and for almost all $x \in \text{int}F$ we have

$$\begin{aligned} f(x) &= \tilde{f}(x) = \int_{\Omega} |\varphi'_x(x, \omega)| \tilde{f}(\varphi(x, \omega)) P(d\omega) + g(x) \\ &= \int_{\Omega} |\varphi'_x(x, \omega)| f(\varphi(x, \omega)) P(d\omega) + g(x). \end{aligned}$$

If $x \in \mathbb{R} \setminus \text{int}F$, then (a) and (5) imply $\varphi(x, \omega) \in \mathbb{R} \setminus \text{int}F$ for almost all $\omega \in \Omega$. Hence $f(x) = 0$ and $f(\varphi(x, \omega)) = 0$ for all $x \in \mathbb{R} \setminus \text{int}F$ and almost all $\omega \in \Omega$. In consequence, (1) holds for almost all $x \in \mathbb{R} \setminus \text{int}F$, by (4).

The proof is complete. \square

4. $L^1(I)$ -SOLUTIONS AS RADON-NIKODYM DERIVATIVES

It is known that integrable solutions of the homogeneous refinement type equation are determined, up to a multiplicative constant, by the Radon-Nikodym derivative of their integrals over $(-\infty, x]$, where $x \in \mathbb{R}$, with respect to the one dimensional Lebesgue measure (see [11, Section 3.4]). We will show that also $L^1(I)$ -solutions of equation (1) have such a property. For this purpose fix $x_0 \in I$ and put

$$\Omega_+ = \{\omega \in \Omega : \varphi'_x(x_0, \omega) > 0\} \quad \text{and} \quad \Omega_- = \{\omega \in \Omega : \varphi'_x(x_0, \omega) < 0\}.$$

From assumption (a) we see that the sets Ω_+ and Ω_- do not depend on the choice of x_0 and $\Omega = \Omega_+ \cup \Omega_-$. Next define a function $G: I \rightarrow \mathbb{R}$ putting

$$(6) \quad G(x) = \int_{\inf I}^x g(t) dt.$$

The following extension of Proposition 3.1 from [10] is a useful tool for studying the existence of $L^1(I)$ -solutions of equation (1).

Lemma 4.1. *Assume $f \in L^1(I)$ and put $\alpha = \int_I f(x) dx$.*

(i) *If $F: I \rightarrow \mathbb{R}$ given by*

$$(7) \quad F(x) = \int_{\inf I}^x f(t) dt$$

satisfies

$$(8) \quad F(x) = \int_{\Omega_+} F(\varphi(x, \omega)) P(d\omega) + \int_{\Omega_-} [\alpha - F(\varphi(x, \omega))] P(d\omega) + G(x)$$

for every $x \in I$, then f is an $L^1(I)$ -solution of equation (1).

- (ii) If f is an $L^1(I)$ -solution of equation (1), then the function $F: I \rightarrow \mathbb{R}$ given by (7) satisfies (8) for every $x \in I$.

Proof. (i) Assume that $F: I \rightarrow \mathbb{R}$ given by (7) satisfies (8) for every $x \in I$. Then

$$\begin{aligned}
 \int_{\inf I}^x f(t)dt &= F(x) = \int_{\Omega_+} \int_{\inf I}^{\varphi(x,\omega)} f(t)dt P(d\omega) \\
 &\quad + \int_{\Omega_-} \left[\int_{\inf I}^{\sup I} f(t)dt - \int_{\inf I}^{\varphi(x,\omega)} f(t)dt \right] P(d\omega) + G(x) \\
 &= \int_{\Omega_+} \int_{\inf I}^{\varphi(x,\omega)} f(t)dt P(d\omega) + \int_{\Omega_-} \int_{\varphi(x,\omega)}^{\sup I} f(t)dt P(d\omega) + G(x) \\
 &= \int_{\Omega_+} \int_{\inf I}^x \varphi'_x(t, \omega) f(\varphi(t, \omega)) dt P(d\omega) \\
 &\quad - \int_{\Omega_-} \int_{\inf I}^x \varphi'_x(t, \omega) f(\varphi(t, \omega)) dt P(d\omega) + G(x) \\
 &= \int_{\Omega} \int_{\inf I}^x |\varphi'_x(t, \omega)| f(\varphi(t, \omega)) dt P(d\omega) + \int_{\inf I}^x g(t) dt \\
 &= \int_{\inf I}^x \left[\int_{\Omega} |\varphi'_x(t, \omega)| f(\varphi(t, \omega)) P(d\omega) + g(t) \right] dt
 \end{aligned}$$

for every $x \in I$. Hence f coincides, in L^1 sense, with $\int_{\Omega} |\varphi'_x(\cdot, \omega)| f(\varphi(\cdot, \omega)) P(d\omega) + g$. This means that f is an $L^1(I)$ -solution of equation (1).

(ii) If f is an $L^1(I)$ -solution of equation (1), then arguing as above we can show that (8) holds with $F: I \rightarrow \mathbb{R}$ defined by (7). \square

5. $L^1(I)$ -SOLUTIONS IN THE CASE WHERE $P(\Omega_+) \in \{0, 1\}$

In this section we will use a result on the convergence in law of a sequence of iterates of a random-valued function obtained in [2]. Thus we assume that there exists $l \in (0, 1)$ such that

$$(9) \quad \int_{\Omega} |\varphi(x, \omega) - \varphi(y, \omega)| P(d\omega) \leq l|x - y| \quad \text{for all } x, y \in I$$

and

$$(10) \quad \int_{\Omega} |\varphi(x, \omega) - x| P(d\omega) < +\infty \quad \text{for every } x \in I.$$

Note that (9) and (a) imply unboundedness of the interval I .

Assume that $\inf I \in \mathbb{R}$ and $\sup I = +\infty$. Then, by (a), we have $\lim_{x \rightarrow \inf I} \varphi(x, \omega) \in \{\inf I, +\infty\}$ for every $\omega \in \Omega$. Put

$$A = \left\{ \omega \in \Omega : \lim_{x \rightarrow \inf I} \varphi(x, \omega) = +\infty \right\}.$$

Fixing $y \in I$ and making use of the Lebesgue monotone convergence theorem and (9) we obtain

$$\begin{aligned} \int_A \lim_{x \rightarrow \inf I} |\varphi(x, \omega) - \varphi(y, \omega)| P(d\omega) &= \lim_{x \rightarrow \inf I} \int_A |\varphi(x, \omega) - \varphi(y, \omega)| P(d\omega) \\ &\leq \lim_{x \rightarrow \inf I} \int_{\Omega} |\varphi(x, \omega) - \varphi(y, \omega)| P(d\omega) \\ &\leq l |\inf I - y| < +\infty, \end{aligned}$$

and hence $P(A) = 0$. In a similar way we can prove that $P(\{\omega \in \Omega : \lim_{x \rightarrow \sup I} \varphi(x, \omega) = -\infty\}) = 0$ in the case where $\inf I = -\infty$ and $\sup I \in \mathbb{R}$.

The above calculation shows that there is no loss of generality in assuming that $\lim_{x \rightarrow \inf I} \varphi(x, \omega) \in \mathbb{R}$ for every $\omega \in \Omega$ in the case where $\inf I \in \mathbb{R}$ and $\lim_{x \rightarrow \sup I} \varphi(x, \omega) \in \mathbb{R}$ for every $\omega \in \Omega$ in the case where $\sup I \in \mathbb{R}$.

Define a function $\varphi_0 : \text{cl}I \times \Omega \rightarrow \text{cl}I$ by putting $\varphi_0 = \varphi$ in the case where $I = \mathbb{R}$, or

$$\varphi_0(x, \omega) = \begin{cases} \varphi(x, \omega) & \text{for } (x, \omega) \in I \times \Omega, \\ \lim_{y \rightarrow x} \varphi(y, \omega) & \text{for } (x, \omega) \in (\text{cl}I \setminus I) \times \Omega, \end{cases}$$

in the case where $I \neq \mathbb{R}$. Note that both φ and φ_0 are random-valued functions, and $\varphi_0(x, \omega) \in \text{cl}I \setminus I$ for all $x \in \text{cl}I \setminus I$ and $\omega \in \Omega$. Moreover, (9) and the Fatou lemma imply

$$(11) \quad \int_{\Omega} |\varphi_0(x, \omega) - \varphi_0(y, \omega)| P(d\omega) \leq l|x - y| \quad \text{for all } x, y \in \text{cl}I,$$

which jointly with (10) gives

$$(12) \quad \int_{\Omega} |\varphi_0(x, \omega) - x| P(d\omega) < +\infty \quad \text{for every } x \in \text{cl}I.$$

Denote by $\pi_n(x, \cdot)$ the distribution of $\varphi_0^n(x, \cdot)$, i.e.,

$$\pi_n(x, B) = P^\infty(\{\varpi \in \Omega^\infty : \varphi_0^n(x, \varpi) \in B\})$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $B \in \mathcal{B}(\text{cl}I)$.

The following result on the convergence in law of the sequence $(\varphi_0^n(x, \cdot))_{n \in \mathbb{N}}$ will be our main tool.

Theorem 5.1. (see [2, Theorem 3.1]) *Assume (12) and let (11) hold with some $l \in (0, 1)$. Then there exists a distribution π on $\text{cl}I$ such that for every $x \in \text{cl}I$ the sequence $(\pi_n(x, \cdot))_{n \in \mathbb{N}}$ converges weakly to π .*

To formulate our first result of this section let us denote by D the probability distribution function of the distribution π obtained in Theorem 5.1, i.e.,

$$D(t) = \pi([\inf I, t] \cap \mathbb{R})$$

for every $t \in \text{cl}I$.

Theorem 5.2. Assume $P(\Omega_+) = 1$. Let (10) hold and let (9) be satisfied with some $l \in (0, 1)$. If

$$(13) \quad \int_I g(t)(1 - D(t))dt = 0$$

and if there exists $L \in (0, +\infty)$ such that

$$(14) \quad \left| \int_x^y g(t)dt \right| \leq L|x - y| \quad \text{for all } x, y \in I,$$

then equation (8) has an absolutely continuous solution $F: I \rightarrow \mathbb{R}$.

Moreover, if there exists $f \in L^1(I)$ such that (7) holds for every $x \in I$, then f is an $L^1(I)$ -solution of equation (1).

Proof. We first observe that it is enough to show that equation

$$(15) \quad F(x) = \int_{\Omega} F(\varphi_0(x, \omega))P(d\omega) + G(x),$$

where $G: \text{cl}I \rightarrow \mathbb{R}$ is defined by (6), has an absolutely continuous solution $F: \text{cl}I \rightarrow \mathbb{R}$; indeed, if F is such a solution, then $F|_I$ is an absolutely continuous solution of equation (8).

Fix $x_0 \in \text{cl}I$. We will show that

$$(16) \quad \lim_{n \rightarrow \infty} \int_{\Omega^\infty} \int_{\inf I}^{\varphi_0^n(x_0, \varpi)} g(t)dt P^\infty(d\varpi) = 0.$$

For every $n \in \mathbb{N}$ denote by D_n the probability distribution function of the distribution $\pi_n(x_0, \cdot)$, i.e.,

$$D_n(t) = P^\infty(\{\varpi \in \Omega^\infty : \varphi_0^n(x_0, \varpi) \leq t\})$$

for every $t \in \text{cl}I$.

Applying the Fubini theorem we obtain

$$\begin{aligned} \int_{\Omega^\infty} \int_{\inf I}^{\varphi_0^n(x_0, \varpi)} g(t)dt P^\infty(d\varpi) &= \int_{\{(t, \varpi) \in \text{cl}I \times \Omega^\infty : t < \varphi_0^n(x_0, \varpi)\}} g(t)dt P^\infty(d\varpi) \\ &= \int_I \int_{\{\varpi \in \Omega^\infty : t < \varphi_0^n(x_0, \varpi)\}} g(t)P^\infty(d\varpi)dt \\ &= \int_I g(t)[1 - P^\infty(\{\varpi \in \Omega^\infty : \varphi_0^n(x_0, \varpi) \leq t\})]dt \\ &= \int_I g(t)dt - \int_I g(t)D_n(t)dt \end{aligned}$$

for every $n \in \mathbb{N}$. From Theorem 5.1 we conclude that for every $t \in \text{cl}I$, being a point of continuity of D , we have $\lim_{n \rightarrow \infty} D_n(t) = D(t)$. Hence $\lim_{n \rightarrow \infty} g(t)D_n(t) = g(t)D(t)$ for almost all $t \in \text{cl}I$, and by the Lebesgue dominated convergence theorem and (13) we obtain

$$\lim_{n \rightarrow \infty} \int_I g(t)D_n(t)dt = \int_I g(t)D(t)dt = \int_I g(t)dt.$$

In consequence, (16) holds.

Finally, according to [2, Corollary 4.1 (iii)] equation (15) has an absolutely continuous solution $F: \text{cl}I \rightarrow \mathbb{R}$.

The moreover statement follows from Lemma 4.1. \square

According to [3, Theorem 3.1 (ii)] we have the following counterpart of Theorem 5.2.

Theorem 5.3. *Assume $P(\Omega_+) = 0$. Let (10) hold and let (9) be satisfied with some $l \in (0, 1)$. If there exists $L \in (0, +\infty)$ such that (14) holds, then equation (8) has an absolutely continuous solution $F: I \rightarrow \mathbb{R}$.*

Moreover, if there exists $f \in L^1(I)$ such that $\int_I f(x)dx = \alpha$ and (7) holds for every $x \in I$, then f is an $L^1(I)$ -solution of equation (1).

At the end of this paper it is worth noting that if an absolutely continuous function $F: I \rightarrow [0, 1]$ is nondecreasing, then it has a Radon-Nikodym derivative. However, not all absolutely continuous real functions have Radon-Nikodym derivatives (see [13, Example 3.4]).

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